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Degree-Based Numerical Invariants of Grasmere Geometric Graph

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Abstract

Graph theory plays a significant role in the applications of chemistry, pharmacy, communication, maps and aeronautical fields. This work is motivated by the old English tiles design on geometric structures, namely grasmere geometric tiles which are very popular in Victorian mosaic tiles. These are French manufactured tiles that have four different colours in it with dogtooth border. In this proposed work, the degree-based topological indices viz., Sombor index, Geometric-Harmonic index, Harmonic-Geometric index, SS index, $RABC$ index, $RABC_4$ index, neighborhood first Zagreb index, neighborhood second Zagreb index, neighborhood hyper Zagreb index, neighborhood third NDe index, neighborhood redefined first Zagreb index, neighborhood redefined second Zagreb index, generalized reciprocal Sanskruti index, neighborhood Sombor index and neighborhood SS index values are computed for grasmere geometric graph.

Keywords: Grasmere Geometric graph, topological indices.

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1. Introduction

Graph theory is widely used in real world communication systems. Precisely, the internet in the whole world is an example of a complex network in graph theory [1, 4, 8]. The complete communication system is an example of graphs in the back end. Not only in the internet, but also graphs play a significant role in biology, mathematical chemistry, logistics, social sciences and so many other fields [3, 15, 18]. All these are connected with a single term is known as network.

Graph theory is a multidisciplinary field known as molecular topology and it relates chemical mathematical models to get insight into the physical and chemical characteristics and bio-activities of chemical compounds [10]. Degree-based molecular descriptors play a significant role specially in chemistry.

In this paper, the grasmere geometric graph [21] is considered for computing the various topological indices. Topological index is also known as connectivity index or numerical invariant. A single square of grasmere geometric graph consists of a combination of triangles and squares, inspired by the design of the French tiles. Because of its unique pattern of geometric designs, it is suitable for the construction of manors, big halls, and exterior pathways.

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Consider a graph $G = (V(G), E(G))$, where $V(G)$ represents vertices and $E(G)$ represents edges of G. The $d_G(b)$ is the degree of a vertex, is the combined sum of vertices and edges depending on the sum of neighborhood degree. The group of all vertices close to b are often called as open neighborhood of b $(N_G(b))$. The closed neighborhood of b is the set $N_G[b] = N_G(a) \cup \{b\}$. The set $N_G[b]$ is the set of closed neighborhood vertices of b. Let $D_G(b) = d_G(b) + \sum_{a \in N_G(b)} d_G(b)$ be the degree sum of closed neighborhood vertices of b.

2. Materials and Methods

The materials used are [6, 11, 16, 17] as this work is purely theoretical. To compute the results the methods used are combinatorial computing, vertex partition method, edge partition method and graph theoretical tools.

Definition 2.1. I. Gutman defined Sombor index [5] as,

$$SO(G) = \sum_{ab \in E(G)} \sqrt{(d_G(a))^2 + (d_G(b))^2}$$

Definition 2.2. A. Usha et al., [19] defined Geometric-Harmonic index as,

$$GH(G) = \sum_{ab \in E(G)} \frac{(d_G(a) + d_G(b))(\sqrt{d_G(a) + d_G(b)})}{2}$$

Definition 2.3. M.C. Shanmukha et al., [14] defined Harmonic-Geometric index as,

$$HG(G) = \sum_{ab \in E(G)} \frac{2}{(d_G(a) + d_G(b))(\sqrt{d_G(a) + d_G(b)})}$$

Definition 2.4. Weidong Zhao et al., [20] defined SS index as,

$$SS(G) = \sum_{ab \in E(G)} \sqrt{\frac{d_G(a) \times d_G(b)}{d_G(a) + d_G(b)}}$$

Definition 2.5. In continuation to SS index, the neighborhood version of SS index is defined as,

$$NSS(G) = \sum_{ab \in E(G)} \sqrt{\frac{S_G(a) \times S_G(b)}{S_G(a) + S_G(b)}}$$

Definition 2.6. Anil Kumar et al., [2] introduced the reciprocal ABC and ABC_4 indices and are defined as,

$$RABC(G) = \sum_{ab \in E(G)} \sqrt{\frac{d_G(a) \times d_G(b)}{d_G(a) + d_G(b) - 2}}$$

$$RABC_4(G) = \sum_{ab \in E(G)} \sqrt{\frac{S_G(a) \times S_G(b)}{S_G(a) + S_G(b) - 2}}$$

Definition 2.7. Sourav Mondal et al., [9] introduced neighborhood version of the first Zagreb index, second Zagreb index, hyper Zagreb index and third NDe index and are defined as,

$$NM_1(G) = \sum_{a \in V(G)} S_G(a)^2 = \sum_{ab \in E(G)} [S_G(a) + S_G(b)]$$

$$NM_2(G) = \sum_{ab \in E(G)} [S_G(a) \times S_G(b)]$$

$$HM_N(G) = \sum_{ab \in E(G)} [S_G(a) + S_G(b)]^2$$

$$ND_3(G) = \sum_{ab \in E(G)} [S_G(a) \times S_G(b)][S_G(a) + S_G(b)]$$

Definition 2.8. Shanmukha et al., [13] introduced neighborhood version of the redefined first and second Zagreb indices and are defined as,

$$NReZ_1(G) = \sum_{ab \in E(G)} \frac{[S_G(a) + S_G(b)]}{[S_G(a) \times S_G(b)]}$$

$$NReZ_2(G) = \sum_{ab \in E(G)} \frac{[S_G(a) \times S_G(b)]}{[S_G(a) + S_G(b)]}$$

Definition 2.9. Shanmukha et al., [12] introduced neighborhood version of the generalized reciprocal Sanskruti index and is defined as,

$$RS(G) = \sum_{ab \in E(G)} \left(\frac{S_G(a) + S_G(b) - 2}{S_G(a) \times S_G(b)} \right)^3$$

Definition 2.10. V.R. Kulli defined neighborhood Sombor index [7] as,

$$NSO(G) = \sum_{ab \in E(G)} \sqrt{(S_G(a))^2 + (S_G(b))^2}$$

3. Grasmere Geometric Graph

To obtain the results, the vertices and edges are partitioned. Specifically, the partitioning of vertex of grasmere geometric graph depends on each vertex degree and is represented in Table 1 and the partitioning of edges of grasmere geometric graph depending on degrees sum and its neighbour degrees sum of end vertices of each edge are represented in Table 2 and Table 3. Here p represents number of rows while q represents the number of columns of grasmere geometric graph.

Table 1: The vertex partition of grasmere geometric graph

$D_G(a) \backslash a \in V(G)$	No. of vertices	Set of vertices
3	$4pq + 4q + 4p + 8$	V_1
4	$4pq + 6q + 6p + 8$	V_2
5	$2q + 2p$	V_3
6	$2pq + 2q + 2p$	V_4
7	pq	V_5

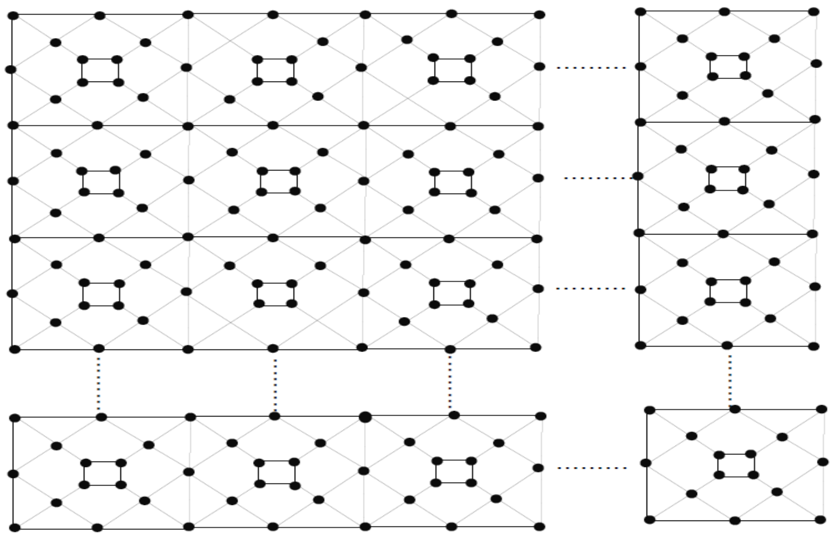

Figure 1: Grasmere geometric graph

Table 2: The edge partition of grasmere geometric graph based on degree sum of end vertices of each edge

(d_a, d_b), where $ab \in E(G)$	No. of edges	(d_a, d_b), where $ab \in E(G)$	No. of edges
$(3,3)$	$4pq+4q+4p+4$	$(4,6)$	$8pq+4q+4p$
$(3,4)$	$4pq+4q+4p+16$	$(4,8)$	$4pq$
$(4,4)$	$4q+4p+8$	$(5,6)$	$2q+2p$
$(4,5)$	$8q+8p$	$(6,8)$	$4pq$

Theorem 3.1. *The Sombor index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{71543}{500}pq + \frac{19411}{125}p + \frac{19411}{125}q + \frac{7111}{50}.$$

Proof. The edges of grasmere geometric graph is denoted by $e_{i,j}$ with $i = d_a$ and $j = d_b$. It is observed from Fig. 1, that the number of edges of the grasmere geometric graph is $24pq + 26q + 26p + 28$, and is the summation of degrees of end points of each edge of grasmere geometric graph have 8 edge types $e_{(3,3)}$, $e_{(3,4)}$, $e_{(4,4)}$, $e_{(4,5)}$,......., $e_{(6,8)}$ as shown in the Table 2.

$$SO(G) = \sum_{ab\in E(G)} \sqrt{(d_G(a))^2 + (d_G(b))^2}$$

$$\begin{aligned}
SO(G) &= e_{(3,3)}(\sqrt{(3)^2+(3)^2}) + e_{(3,4)}(\sqrt{(3)^2+(4)^2}) + e_{(4,4)}(\sqrt{(4)^2+(4)^2}) + e_{(4,5)}(\sqrt{(4)^2+(5)^2}) \\
&\quad + e_{(4,6)}(\sqrt{(4)^2+(6)^2}) + e_{(4,8)}(\sqrt{(4)^2+(8)^2}) + e_{(5,6)}(\sqrt{(5)^2+(6)^2}) + e_{(6,8)}(\sqrt{(6)^2+(8)^2}) \\
&= (4pq+4q+4p+4)(\sqrt{18}) + (4pq+4q+4p+16)(5) + (4q+4p+8)(\sqrt{32}) + (8q+8p)(\sqrt{41}) \\
&\quad + (8pq+4q+4p)(\sqrt{52}) + (4pq)(\sqrt{80}) + (2q+2p)(\sqrt{61}) + (4pq)(\sqrt{10}) \\
SO(G) &= \frac{71543}{500}pq + \frac{19411}{125}p + \frac{19411}{125}q + \frac{7111}{50}.
\end{aligned}$$

□

Theorem 3.2. *The Geometric-Harmonic index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{305105}{500}pq + \frac{431723}{1000}p + \frac{431723}{1000}q + \frac{35799}{100}.$$

Proof. From Table 2, using the number of respective edges and from the definition of Geometric-Harmonic index, we get

$$GH(G) = \frac{305105}{500}pq + \frac{431723}{1000}p + \frac{431723}{1000}q + \frac{35799}{100}.$$

□

Theorem 3.3. *The Harmonic-Geometric index of grasmere geometric graph for $p > 1$ and $q > 1$ is*

$$\frac{1301}{1000}pq + \frac{413}{250}p + \frac{413}{250}q + \frac{283}{125}.$$

Proof. From Table 2, using the number of respective edges and from the definition of Harmonic-Geometric index, we get

$$HG(G) = \frac{1301}{1000}pq + \frac{413}{250}p + \frac{413}{250}q + \frac{283}{125}.$$

□

Theorem 3.4. *The SS index of grasmere geometric graph for $p > 1$ and $q > 1$ is*

$$\frac{7336}{125}pq + \frac{18609}{500}p + \frac{18609}{500}q + \frac{18581}{500}.$$

Proof. From Table 2, using the number of respective edges and from the definition of SS index, we get

$$SS(G) = \frac{7336}{125}pq + \frac{18609}{500}p + \frac{18609}{500}q + \frac{18581}{500}.$$

□

Theorem 3.5. *The RABC index of grasmere geometric graph for $p > 1$ and $q > 1$ is*

$$\frac{5151}{125}pq + \frac{42831}{1000}p + \frac{42831}{1000}q + \frac{43851}{1000}.$$

Proof. From Table 2, using the number of respective edges and from the definition of RABC index, we get

$$RABC(G) = \frac{5151}{125}pq + \frac{42831}{1000}p + \frac{42831}{1000}q + \frac{43851}{1000}.$$

□

Table 3: The edge partition of grasmere geometric graph based on neighbor degree sum of end vertices of each edge

(S_a, S_b), where $ab \in E(G)$	No. of edges	(S_a, S_b), where $ab \in E(G)$	No. of edges
$(10,10)$	$4pq+4q+4p+4$	$(18,18)$	$4q+4p-8$
$(10,14)$	4	$(18,22)$	$8q+8p-8$
$(10,18)$	$4q+4p$	$(18,26)$	0
$(10,23)$	$4pq$	$(18,29)$	$4q+4p$
$(12,14)$	4	$(22,26)$	0
$(12,16)$	8	$(22,29)$	$2q+2p$
$(14,14)$	0	$(23,29)$	$4q+4p$
$(14,16)$	8	$(23,32)$	$8pq-4q-4p$
$(14,18)$	0	$(23,40)$	$4pq$
$(14,22)$	0	$(29,40)$	$2q+2p$
$(16,18)$	8	$(32,40)$	$4pq-2q-2p$
$(16,22)$	8	−	−

Theorem 3.6. *The neighborhood version of the first Zagreb index of grasmere geometric Graph for* $p > 1$ *and* $q > 1$ *is*

$$1192pq + 1364p + 1364q + 712.$$

Proof. The edges of grasmere geometric graph is denoted by $e_{i,j}$ with $i = S_a$ and $j = S_b$. It is observed from Fig. 1, that the number of edges of the grasmere geometric graph is $24pq + 26q + 26p + 28$, and is the summation of neighborhood degrees of end points of each edge of grasmere geometric graph have 23 edge types $e_{(10,10)}$, $e_{(10,14)}$, $e_{(10,18)}$, $e_{(10,23)}$,......, $e_{(32,40)}$ as shown in the Table 3.
The neighborhood version of the first Zagreb index

$$NM_1(G) = \sum_{ab \in E(G)} [S_G(a) + S_G(b)]$$

$$\begin{aligned}
NM_1(G) &= e_{(10,10)}(10+10) + e_{(10,14)}(10+14) + e_{(10,18)}(10+18) \\
&+ e_{(10,23)}(10+23) + e_{(12,14)}(12+14) + e_{(12,16)}(12+16) + e_{(14,16)}(14+16) \\
&+ e_{(16,18)}(16+18) + e_{(16,22)}(16+22) + e_{(18,18)}(18+18) + e_{(18,22)}(18+22) \\
&+ e_{(18,29)}(18+29) + e_{(22,29)}(22+29) + e_{(23,29)}(23+29) + e_{(23,32)}(23+32) \\
&+ e_{(23,40)}(23+40) + e_{(29,40)}(29+40) + e_{(32,40)}(32+40) \\
&= (4pq + 4q + 4p + 4)(100) + (4)(140) + (4q + 4p)(180) + (4pq)(230) \\
&+ (4)(168) + (8)(192) + (8)(224) + (8)(288) + (8)(352) + (4q + 4p - 8)(324) \\
&+ (8q + 8p - 8)(396) + (4q + 4p)(522) + (2p + 2q)(638) + (4q + 4p)(667) \\
&+ (8pq - 4q - 4p)(736) + (4pq)(920) + (2q + 2p)(1160) + (4pq - 2q - 2p)(1280) \\
NM_1(G) &= 1192pq + 1364p + 1364q + 712.
\end{aligned}$$

□

Theorem 3.7. *The neighborhood version of the second Zagreb index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$16008pq + 5872p + 5872q + 4320.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the second Zagreb index, we get

$$NM_2(G) = 16008pq + 5872p + 5872q + 4320.$$

□

Theorem 3.8. *The neighborhood version of the hyper Zagreb index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$66768pq + 24260p + 24260q + 17712.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the hyper Zagreb index, we get

$$HM_N(G) = 66768pq + 24260p + 24260q + 17712.$$

□

Theorem 3.9. *The neighborhood version of the third NDe index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$962680pq + 133004p + 133004q + 100992.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the third NDe index, we get

$$ND_3(G) = 962680pq + 133004p + 133004q + 100992.$$

□

Theorem 3.10. *The neighborhood version of the redefined first Zagreb index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{12353}{5000}pq + \frac{1551}{500}p + \frac{1551}{500}q + \frac{2227}{500}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the redefined first Zagreb index, we get

$$NReZ_1(G) = \frac{12353}{5000}pq + \frac{1551}{500}p + \frac{1551}{500}q + \frac{2227}{500}.$$

□

Theorem 3.11. *The neighborhood version of the redefined second Zagreb index of grasmere geometric Graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{14223}{50}pq + \frac{47663}{250}p + \frac{47663}{250}q + \frac{6341}{25}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the redefined second Zagreb index, we get

$$NReZ_2(G) = \frac{14223}{50}pq + \frac{47663}{250}p + \frac{47663}{250}q + \frac{6341}{25}.$$

□

Theorem 3.12. *The generalized reciprocal Sanskruti index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{95}{2500}pq + \frac{255}{5000}p + \frac{255}{5000}q + \frac{135}{1250}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the generalized reciprocal Sanskruti index, we get

$$RS(G) = \frac{95}{2500}pq + \frac{255}{5000}p + \frac{255}{5000}q + \frac{135}{1250}.$$

□

Theorem 3.13. *The neighborhood Sombor index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{43081}{50}pq + \frac{332137}{500}p + \frac{332137}{500}q + \frac{101693}{200}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the Sombor index, we get

$$NSO(G) = \frac{43081}{50}pq + \frac{332137}{500}p + \frac{332137}{500}q + \frac{101693}{200}.$$

□

Theorem 3.14. *The neighborhood SS index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{2023}{25}pq + \frac{1903}{25}p + \frac{1903}{25}q + \frac{1751}{25}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the SS index, we get

$$NSS(G) = \frac{2023}{25}pq + \frac{1903}{25}p + \frac{1903}{25}q + \frac{1751}{25}.$$

□

Theorem 3.15. *The* $RABC_4$ *index of grasmere geometric graph for* $p > 1$ *and* $q > 1$ *is*

$$\frac{10343}{125}pq + \frac{90727}{1000}p + \frac{90727}{1000}q + \frac{72963}{1000}.$$

Proof. From Table 3, using the number of respective edges and from the definition of neighborhood version of the SS index, we get

$$RABC_4(G) = \frac{10343}{125}pq + \frac{90727}{1000}p + \frac{90727}{1000}q + \frac{72963}{1000}.$$

□

4. CONCLUSION

In this paper, degree-based topological indices viz., Sombor index, GH index, HG index, SS index, RABC index, $RABC_4$ index, neighborhood first Zagreb index, neighborhood second Zagreb index, neighborhood hyper Zagreb index, neighborhood third NDe index, neighborhood redefined first Zagreb index, neighborhood redefined second Zagreb index, generalized reciprocal Sanskruti index, neighborhood Sombor index and neighborhood SS index values are computed for grasmere geometric graph. The results give a phenomenal contribution to the graph theory and also to complex networks.

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